# Recent topics on Monochromatic Structures in Edge-colored Graphs (Designs, Codes, Graphs and Related Areas)



藤田, 慎也

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# Recent topics on Monochromatic Structures in Edge-colored Graphs

SHINYA FUJITA*

International College of Arts and Sciences
Yokohama City University
22-2, Seto, Kanazawa-ku, Yokohama, 236-0027,
Japan
fujita@yokohama-cu.ac.jp

**Abstract:** We will review some recent results on the existence of monochromatic subgraphs with certain properties in edge-colored graphs.

## 1 Introduction

We consider only finite and simple graphs. In particular, we will mainly consider edge-colored graphs. Given a graph whose edges are colored, on how many vertices can we find a monochromatic subgraph of a certain type, such as a connected subgraph, or a cycle? In this short survey, we shall review some known results and conjectures regarding these questions.

We firstly give some basic definitions. For a graph $G=(V(G),E(G))$, let $c(G)$ be the *circumference* of $G$, i.e. the length of a longest cycle in $G$. Let $\alpha(G)$ be the independence number of $G$, i,e., the size of the largest independent set of $G$. For two disjoint graphs $A$ and $B$, let $A+B$ be the graph obtained from $A$ and $B$ by joining them completely with edges (thus, $V(A+B)=V(A)\cup V(B), E(A+B)= E(A)\cup E(B)\cup\{ab|a\in V(A), b\in V(B)\})$. A graph $G$ is called *unicyclic* if it has exactly one cycle. Let $P_{4}^{+}$ be a $P_{4}$ with the addition of a single vertex adjacent to an internal vertex of the path.

## 2 Monochromatic cycles

In this section, let us consider the problem of finding monochromatic subgraphs in edge-colored graphs. A first result in this direction is the following observation, made a long time ago by Erdős and Rado: *A graph is either connected, or its complement is connected.* In other words, for every 2-edge-colored complete graph, there exists a monochromatic spanning connected subgraph (or equivalently, a monochromatic spanning tree). A substantial generalization of this observation is to ask for the existence of a large monochromatic subgraph of a certain type in an edge-colored graph.

Given an $r$-edge-colored complete graph, we may ask for the existence of a long monochromatic cycle. Throughout this section we regard $K_{i}$ as a cycle of order $i$ for $i\in\{1,2\}$. Let us consider the following problem:

**Problem 1** *Determine the maximum value $f(n,r)$ such that every $r$-edge-coloring of $K_{n}$ contains a monochromatic cycle of length at least $f(n,r)$.*

In [6] Faudree et al. showed that for every graph $G$ of order $n\geq 6$ we have $\max\{c(G), c(\overline{G})\}\geq\lceil 2n/3\rceil$, where $\overline{G}$ denotes the complement of $G$. Furthermore, this bound is sharp. It can be easily seen by taking $G$ to be the graph consisting of $\lfloor n/3\rfloor$ isolated vertices and a clique on the remaining $\lceil 2n/3\rceil$ vertices. So we have $f(n,2)=\lceil 2n/3\rceil$. For $r\geq 3$, it is known that $f(n,r)\leq n/(r-1)$.

The lower bound on $f(n,r)$ is given as follows:

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**Theorem 2** ([7]) *Let $n, r$ be integers with $n \geq r \geq 1$. Then any $r$-edge-colored complete graph $K_n$ contains a monochromatic cycle of order at least $\lceil n/r \rceil$. (i.e., $f(n,r) \geq \lceil n/r \rceil$. )*

Very recently, Theorem 2 was slightly improved in some special cases:

**Theorem 3** ([10]) *Let $n, r$ be integers with $n \geq r \geq 1$. Suppose that both $n$ and $\lceil \frac{n(n-1)-2r}{(n-2)r} \rceil$ are even. Then any $r$-edge-colored complete graph $K_n$ contains a monochromatic cycle of order at least $\lceil \frac{n(n-1)-2r}{(n-2)r} \rceil$.*

Another recent progress on this problem is the following:

**Theorem 4** ([11]) *The following statements hold:*

(*i*) *For $n \geq r \geq 3$, $f(2r+2, r) = 3$.*

(*ii*) *For any positive integers $s, c$ with $s \geq 2, c \geq 2$, $f(sr + c, r) = s + 1$ holds if $r$ is sufficiently large compared with $s$ and $c$.*

This theorem says that there exist infinitely many pairs $n, r$ such that $f(n,r) = \lceil n/r \rceil$. But we do not know the exact value of $f(n,r)$ in other cases. Even for the case $f(n,3)$, it is open.

# 3 Gallai-colorings and extensions

In this topic, we shall consider the task of finding monochromatic subgraphs in edge-colored complete graphs by putting a restriction on the edge-coloring. Edge colorings of complete graphs in which no triangle is colored with three distinct colors were called Gallai-partitions in [25], and Gallai-colorings in [20, 21]. Here we briefly call these colorings $G$-colorings and always assume that $G$-colorings are on the edges of a complete graph. More than just the term, the concept occurs in relation to deep structural properties of fundamental objects. An important result, Theorem 5, from Gallai's original paper [17] -translated to English and endowed by comments in [26] - can be reformulated in terms of $G$-colorings. Further occurrences are related to generalizations of the perfect graph theorem [2, 3], Ramsey-type functions called *Gallai-Ramsey numbers* [13, 16], or applications in information theory [24].

Our starting point in this section is the following result of Gallai [17], see an explicit proof in [20]. We say that a color class of an edge-coloring of $G$ is *connected* if it together with all vertices of $G$ forms a connected graph. Otherwise the color class is called *disconnected.*

**Theorem 5** *In every $G$-coloring with at least three colors, at least one of the color classes must be disconnected.*

What is the role of forbidding a rainbow triangle? Call a subgraph *rainbow* if all colors on the edges of the subgraph are distinct. Can we extend Theorem 5 in some way to colorings where a rainbow copy of some other fixed graph $F$ is forbidden? This question is the central topic of this section. An edge coloring of a complete graph $K$ is *connected* if every color class in $K$ is connected. Let us say that a graph $F$ has the *disconnection property*, $DP$, if there exists a natural number $m = m(F)$ (note that $m(F)$ does not depend on the order of $K$) such that the following holds: in every edge coloring of a complete graph with at least $m$ colors, either there is a rainbow $F$ or at least one color class is disconnected. Equivalently, $F$ has the disconnection property if, in every connected coloring with at least $m(F)$ colors, there is a rainbow copy of $F$. Notice that $m(F) \geq |E(F)|$ because complete graphs which are large enough have connected colorings using $|E(F)| - 1$ colors with no rainbow $F$.

By definition, Theorem 5 tells us that $K_3 \in DP$. In [12] $K_1 + (K_1 \cup K_2) \in DP$ is shown. The recent progress on this topic is the following:

**Theorem 6** ([9]) *The following statements hold:*

(*i*) *If $F \in DP$ is connected and bipartite, then $F$ is a tree or a unicyclic graph or two such components joined by an edge.*

(*ii*) *For any* $F \in DP$, *there exists an edge* $e \in E(F)$ *such that* $F - e$ *is bipartite.*

(*iii*) *If* $F \in DP$ *is connected, then* $F$ *can be obtained from a tree by adding at most two edges.*

(*iv*) *If* $F$ *is a unicyclic graph such that its cycle is a triangle, then* $F \in DP$. (*hence, any forest belongs to* $DP$.)

We do not know whether small cycles with at least 4 vertices are in $DP$. So we propose the following problem:

**Problem 7** *Is* $C_4 \in DP$? *More generally, are even cycles in* $DP$?

In [9] the authors construct an example which shows that if $C_4 \in DP$ then $m(C_4) > 4(= |E(C_4)|)$.

## 4 Covering by monochromatic subgraphs and related topics

So far, much work has been done on covering problems in edge-colored complete graphs. Those come from a variety of background, but mostly the purpose in this topic is to cover the whole vertex set of $K_n$ by monochromatic connected components. One such example is the following, which is the equivalent formulation of the Ryser's conjecture on multi-partite hypergraphs [22, 27]:

**Conjecture 8** *In every* $r$*-edge-coloring of a complete graph, the vertex set can be covered by the vertices of at most* $r - 1$ *monochromatic connected components.*

This conjecture is open for $r \geq 6$. It is trivially true for $r = 2$, the cases $r = 3, 4$ are solved in [18] and in [5], and for the case $r = 5$, see [5, 28].

Gyárfás and Lehel discovered a bipartite version of this conjecture.

**Conjecture 9** *In every* $r$*-edge-coloring of a complete bipartite graph, the vertex set can be covered by the vertices of at most* $2r - 2$ *monochromatic connected components.*

It is easy to check that any $r$-edge-coloring of a complete bipartite graph contains at most $2r - 1$ monochromatic connected components covering the whole vertex set. Indeed, let $u$ and $v$ be two vertices in opposite classes of $K_{m,n}$, and take the monochromatic double star with centers $u$ and $v$, along with the remaining monochromatic stars centered at $u$ and $v$ (there are at most $2r - 2$ such stars). On the other hand, it is shown in [4] that there is an $r$-edge-coloring of a complete bipartite graph where we need at least $2r - 2$ monochromatic connected components to cover the vertex set.

The recent progress on this conjecture is the following:

**Theorem 10 ([4])** *Conjecture 9 is true for* $r \leq 5$.

We now give a quick review concerning the existence of large monochromatic trees in edge-colored graphs with given independence number. In [19], Gyárfás and Sárközy investigated the size of monochromatic trees in edge-colored graphs.

**Theorem 11 ([19])** *Any* 2*-edge-colored graph* $G$ *contains a monochromatic tree* $T$ *of order at least* $|V(G)|/\alpha(G)$.

**Theorem 12 ([19])** *Any* $G$*-colored graph* $G$ *contains a monochromatic tree* $T$ *of order at least* $|V(G)|/(\alpha(G)^2 + \alpha(G) - 1)$.

The bound on $T$ in Theorem 11 is sharp. To see this, consider $\alpha(G)$ disjoint monchromatic complete graphs of equal order. We do not know about the best possiblity on the order of $T$ in Theorem 12.

Recently, Theorem 11 was extended to a result on partitioning $V(G)$ by monochromatic connected subgraphs.

**Theorem 13** ([8]) *Any 2-edge-colored graph $G$ can be partitioned into at most $\alpha(G)$ monochromatic connected parts.*

Now we consider another different covering problem concerning highly connected monochromatic subgraphs in edge-colored complete graphs. Returning to the case $r = 2$ in Conjecture 8, we see that any 2-coloring of $K_n$ is covered by a monochromatic connected subgraph. However, when we try to find such a subgraph with higher connectivity, we can not hope to find such a spanning subgraph. In order to see this, consider the following example:

Let $G_n = H_1 \cup \cdots \cup H_5$ where $H_i$ is a red complete graph $K_{k-1}$ for $i \leq 4$ and $H_5$ is a red $K_n - 4(k-1)$ where $n > 4(k-1)$. To this structure, we add all possible red edges between $H_5$, $H_1$ and $H_2$ and from $H_1$ to $H_3$ and from $H_2$ to $H_4$. All edges not already colored in red are colored in blue. In either color, there is no $k$-connected subgraph of order larger than $n - 2(k-1)$. Since a spanning monochromatic subgraph is more than we could hope for, we consider finding a highly connected subgraph that is as large as possible. Along this line, Bollobás and Gyárfás [1] proposed the following conjecture.

**Conjecture 14** *For $n > 4(k-1)$, every 2-coloring of $K_n$ contains a monochromatic $k$-connected subgraph with at least $n - 2(k-1)$ vertices.*

In order to see that the bound on $n$ is the best possible, consider the example $G_n$ above with $n = 4(k-1)$ (so $H_5 = \emptyset$). In [1], the authors showed that this conjecture is true for $k \leq 2$.

The recent progress concerning Conjecture 14 is the following:

**Theorem 15** ([14]) *If $n > 6.5(k-1)$ then any 2-edge-coloring of $K_n$ contains a monochromatic $k$-connected subgraph of order at least $n - 2(k-1)$.*

By the example $G_n$, we must give up finding a monochromatic $k$-connected subgraph covering the vertex set of a 2-edge-colored $K_n$. But how about covering "almost" all the vertices by a monochromatic $k$-connected subgraph? If $n$ is extremely large compared with $k$, one can say from Theorem 15 that any 2-edge-coloring of $K_n$ contains a monochromatic $k$-connected subgraph which covers "almost" all of the vertices. Can we have a similar statement for any $r$-edge-coloring of $K_n$ with $r \geq 3$? This is not true in general. If we consider an $r$-edge-coloring of $K_n$ and try to find the largest monochromatic $k$-connected subgraph of $K_n$, it was shown in [23] that the best result one could possibly hope for would be a monochromatic $k$-connected subgraph of order approximately $\frac{n}{r-1}$. Thus, in order to find larger monochromatic $k$-connected subgraphs, it becomes necessary to assume additional restrictions on the coloring.

Finding a monochromatic $k$-connected subgraph covering almost all of the vertices corresponds to finding one color class inducing an "almost" $k$-connected graph. In contrast to the concept $DP$ in the previous section, one very natural restriction would be to forbid the existence of a rainbow subgraph.

Thus, we have the following question:

**Problem 16** *Let $n, r, k$ be positive integers with $n \gg r \gg k$. For what connected graphs $G$ does the following statement hold? In any rainbow $G$-free coloring of $K_n$ using at least $r$ colors, there is a monochromatic $k$-connected subgraph of order at least $n - f(G, r, k)$ for some function $f$ not depending on $n$.*

The following result gives an answer toward this question:

**Theorem 17** ([15]) *The set of graphs $G$ such that $G$ satisfies Question 16 is precisely $K_3, P_4^+$ and $P_6$ and their subgraphs.*

## References

[1] B. Bollobás, A. Gyárfás, *Highly connected monochromatic subgraphs*, Discrete Math., **308** (2008), 1722–1725.

[2] K. Cameron, J. Edmonds, *Lambda composition*, J. Graph Theory **26** (1997), 9–16.

[3] K. Cameron, J. Edmonds, L. Lovász, A note on perfect graphs, Periodica Math. Hungar., **17** (1986), 173–175.

[4] G. Chen, S. Fujita, A. Gyárfás, J. Lehel, Á. Tóth, *Around a biclique cover conjecture*, arXiv:1212.6861[math.CO].

[5] P. Duchet, *Représentations, noyaux en théorie des graphes at hypergraphes*, Thesis, Paris 1979.

[6] R. J. Faudree, L. Lesniak, I. Schiermeyer, *On the circumference of a graph and its complement*, Discrete Math., **309** (2009), 5891–5893.

[7] S. Fujita, *Some remarks on long monochromatic cycles in edge-colored complete graphs*, Discrete Math., **311** (2011), 688–689.

[8] S. Fujita, M. Furuya, A. Gyárfás, Á. Tóth, *Partition of graphs and hypergraphs into monochromatic connected parts* , Electronic J. Combin., **19** (2012), #P27.

[9] S. Fujita, A. Gyárfás, C. Magnant, A. Seress, *Disconnected colors in generalized Gallai colorings*, J. Graph Theory, **74** (2013), 104-114.

[10] S. Fujita, L. Lesniak, *Revisit of Erdős-Gallai's theorem on the circumference of a graph*, Information Processing Letters, **113** (2013), 646-648.

[11] S. Fujita, L. Lesniak, Á. Tóth, *Further remarks on long monochromatic cycles in edge-colored complete graphs*, J. Combin. Math. Combin. Comput., In press.

[12] S. Fujita, C. Magnant, *Extensions of Gallai-Ramsey results*, J. Graph Theory, **70** (2012), 404–426.

[13] S. Fujita, C. Magnant, *Gallai Ramsey numbers for cycles*, Discrete Math., **311** (2011), 1247–1254.

[14] S. Fujita, C. Magnant, *Note on highly connected monochromatic subgraphs in 2-colored complete graphs*, Electronic J. Combin., **18** (2011), #P15.

[15] S. Fujita, C. Magnant, *Forbidden rainbow subgraphs that force highly connected monochromatic subgraphs*, SIAM J. on Discrete Math., **27** (2013), 1625-1637.

[16] S. Fujita, C. Magnant, K. Ozeki, *Rainbow generalizations of ramsey theory: A survey*, Graphs and Combin., **26** (2010), 1–30.

[17] T. Gallai, *Transitiv orientierbare Graphen*, Acta Math. Sci. Hungar., **18** (1967), 25–66. English translation in [26].

[18] A. Gyárfás, *Partition coverings and blocking sets in hypergraphs* (in Hungarian), Communications of the Computer and Automation Institute of the Hungarian Academy of Sciences **71** (1977), 62 pp. MR 58, 5392.

[19] A. Gyárfás, G. N. Sárközy, *Gallai colorings of non-complete graphs*, Discrete Math., **310** (2010), 977–980.

[20] A. Gyárfás, G. Simonyi, *Edge colorings of complete graphs without tricolored triangles*, J. Graph Theory **46** (2004), 211–216.

[21] A. Gyárfás, G. N. Sárközy, A. Sebő, S. Selkow, Ramsey-type results for Gallai colorings, J. Graph Theory **64** (2009), 233–243.

[22] J. R. Henderson, *Permutation decomposition of* $(0-1)$*-matrices and decomposition transversals*, Ph.D. thesis, Caltech, 1971.

[23] H. Liu, R. Morris, N. Prince, *Highly connected monochromatic subgraphs of multicolored graphs*, J. Graph Theory **61** (2009), 22–44.

[24] J. Körner, G. Simonyi, *Graph pairs and their entropies: Modularity problems*, Combinatorica **20** (2000), 227–240.

[25] J. Körner, G. Simonyi, Z. Tuza, *Perfect couples of graphs*, Combinatorica **12** (1992), 179–192.

[26] F. Maffray, M. Preissmann, *A Translation of Gallai's Paper: Transitiv Orientierbare Graphen*, in:J. L. Ramirez-Alfonsin and B. A. Reed (editors), Perfect Graphs, John Wiley and Sons, (2001), 25–66.

[27] J. Lehel, *Ryser's conjecture for linear hypergraphs*, manuscript, 1998.

[28] Z. Tuza, *Some special cases of Ryser's conjecture*, unpublished manuscript, 1978.